MTH849 QUALIFYING EXAM - JUNE 2021

1. Instructions

There are five problems on this exam. Complete as many problems as possible. Your four highest scoring answers will be used to determine your grade on the exam. A preference in scoring will be given to complete answers to entire problems, in contrast to partial answers to possibly more problems. You have also been given a pdf document containing various theorems and definitions for your use on this exam.

- There are NO outside materials allowed on this exam. NO personal notes. NO lecture notes. NO textbooks. NO D2L documents. Nothing other than the exam itself and the theorem list provided with the exam.
- There is no collaboration and no interaction with any human or digital source during the exam.
- This is enforced via the honor system.

2. The exam questions

Problem 1. Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded, connected, and has a C^1 boundary. Assume that $a^{ij} \in L^\infty(\Omega)$ is a uniformly elliptic matrix in the sense that there exists two constants $\lambda > 0$ and $\Lambda > 0$ so that

$$\forall\, \xi \in \mathbb{R}^n, \quad \lambda \left|\xi\right|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda \left|\xi\right|^2, \tag{1}$$

and that $c \in L^\infty(\Omega)$ is given. The bilinear form associated to a^{ij} and c is:

$$B[u,v] = \int_\Omega a^{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} dx + \int_\Omega c(x) uv dx.$$

(i) Assume that $c(x) \geq 0$. Prove that B is coercive on $H_0^1(\Omega)$.

(ii) Now you have no assumption on the sign of $c(x)$. Prove that there exists a choice of λ in (1), depending upon Ω, n, and $\|c\|_{L^\infty}$, so that the bilinear form, B, is coercive on $H_0^1(\Omega)$.

Problem 2. Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded, connected, and has a C^1 boundary. You will answer questions about the following Neumann problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases} \tag{2}$$

(i) Assuming that f and g are continuous functions, prove that if there exists $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ that solves (2) classically, then

$$\int_\Omega f dx + \int_{\partial\Omega} g dS = 0.$$

(ii) Define V to be the subspace of H^1 as

$$V = \{v \in H^1(\Omega) \ : \ \int_\Omega v dx = 0\}. \tag{3}$$

Prove that for all $f \in L^2(\Omega)$ and all $g \in L^2(\partial\Omega)$, there exists a unique $v \in V$ that satisfies the following equation:

$$\forall\ w \in V, \quad \int_\Omega \nabla v \cdot \nabla w dx = \int_\Omega f w dx + \int_{\partial\Omega} gTwdS, \tag{4}$$

where $T : H^1(\Omega) \to L^2(\partial\Omega)$ is the trace operator.

(iii) The actual definition of weak solution that is compatible with classical solutions of (2) is that $u \in V$, and

$$\forall\ w \in H^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla w dx = \int_\Omega f w dx + \int_{\partial\Omega} gTwdS, \tag{5}$$

where $T : H^1(\Omega) \to L^2(\partial\Omega)$ is the trace operator.

Identify a restriction on $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ so that given any f and g that satisfy your restriction, there exists a unique $u \in V$ that solves (5). Furthermore, prove that given such an f and g, there does indeed exists a unique $u \in V$ that solves (5).

Problem 3. Assume that $B_1(0) \subset \mathbb{R}^n$ is the unit ball, and that a^{ij} are uniformly elliptic coefficients. You will prove an energy estimate for weak solutions of

$$-\frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial u}{\partial x_i}\right) = f + \frac{\partial}{\partial x_j} g_j \quad \text{in } B_1.$$

When u is a weak solution of the Dirichlet problem, with $u \in H_0^1(B_1)$, i.e.

$$\forall\ v \in H_0^1(B_1), \quad \int_{B_1} a^{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x)dx = \int_{B_1} f v dx - \int_{B_1} g_j \frac{\partial v}{\partial x_j} dx,$$

prove the estimate,

$$\int_{B_1} |\nabla u|^2 \le C\left(\int_{B_1} u^2 dx + \int_{B_1} f^2 dx + \sum_j \int_{B_1} |g_j|^2\right).$$

Problem 4. Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded, connected, and has a C^1 boundary. Further assume that $\alpha \in L^\infty(\Omega)$, with $\alpha \ge 0$, and

$$\int_{\partial\Omega} \alpha dS > 0.$$

Prove that there exists a constant, C, depending on Ω and n, so that

$$\forall\ u \in H^1(\Omega), \quad \int_\Omega u^2 dx \le C\left(\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} \alpha(x)(Tu)^2 dS\right). \tag{6}$$

Problem 5. Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded, connected, with a C^1 boundary. Assume further that $a^{ij} \in L^\infty(\Omega)$ is a uniformly elliptic matrix, and that $\alpha \in L^\infty(\partial\Omega)$ satisfies $\alpha \ge 0$ and

$$\int_\Omega \alpha(x) dS > 0.$$

Consider the Robin problem with coefficients:

$$\begin{cases} -\frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial u}{\partial x_i}\right) = f & \text{in } \Omega \\ \alpha u + a^{ij}\frac{\partial u}{\partial x_i}\nu_j = g & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $\nu = (\nu_j)$ is the outward normal vector to $\partial\Omega$. A weak solution of (7) is a function, $u \in H^1(\Omega)$ that satisfies

$$\forall\ v \in H^1(\Omega), \quad \int_\Omega a^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \int_{\partial\Omega} \alpha(x) Tu Tv dS = \int_\Omega f dx + \int_{\partial\Omega} g Tv dS, \tag{8}$$

where $T : H^1(\Omega) \to L^2(\partial\Omega)$ is the trace operator on Ω.

(i) Prove that given any $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$, there exists a unique $u \in H^1(\Omega)$ that is a weak solution of (7), i.e. u satisfies (8).

(ii) Assume that $f \in C^0(\Omega)$, $a^{ij} \in C^0(\Omega)$, $\alpha, g \in C^0(\partial\Omega)$, and $u \in H^1(\Omega)$ is the unique weak solution that solves (8). Prove that if additionally, $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, then for all $x \in \overline{\Omega}$, u satisfies (7) pointwise.